A point P in the plane of a triangle

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Find a point P in the plane of a given triangle ABC such that the sum

$$\frac{|AP|^2}{b^2}+\frac{|BP|^2}{c^2}+\frac{|CP|^2}{a^2}$$

is minimal, where $a=BC, b=CA, c=AB$.

Solution by Arkady Alt , San Jose, California, USA.

Let Q be point on the plane with barycentric coordinates $(q_a,q_b,q_c)=\left(\frac{1}{kb^2},\frac{1}{kc^2},\frac{1}{ka^2}\right)$, where $k=\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a^2}$. Then by Leibnitz Formula for distance between two points in barycentric coordinates we obtain

$$PQ^2=\sum_{cyc}q_aPA^2-\sum_{cyc}q_bq_ca^2=\frac{1}{k}\sum_{cyc}\frac{PA^2}{b^2}-\frac{1}{k^2}\sum_{cyc}\frac{1}{c^2a^2}\cdot a^2=$$

$$\frac{1}{k}\sum_{cyc}\frac{PA^2}{b^2}-\frac{1}{k^2}\sum_{cyc}\frac{1}{c^2}=\frac{1}{k}\left(\sum_{cyc}\frac{PA^2}{b^2}-1\right).$$

Hence, $\sum_{cyc}\frac{PA^2}{b^2}=k\cdot PQ^2+1$ and, therefore, $\min\sum_{cyc}\frac{PA^2}{b^2}=1=\sum_{cyc}\frac{QA^2}{b^2}$.

That is $\sum_{cyc}\frac{PA^2}{b^2}$ is minimal iff $P=Q$, where Q is intersect point of cevians AA_1,BB_1,CC_1 such that $\frac{BA_1}{A_1C}=\frac{F_c}{F_b}=\frac{q_c}{q_b}=\frac{c^2}{a^2},\frac{CB_1}{B_1A}=\frac{q_a}{q_c}=\frac{a^2}{b^2}$, $\frac{AC_1}{C_1B}=\frac{q_b}{q_a}=\frac{b^2}{c^2}$.